Mathematical Modelling and Analysis I
Individual Project

This document studies the water balance in the great lakes

MATHEMATICAL MODELLING

It can be understood that the water balance in a lake can be expressed as

$\frac{dV}{dt}$ = flow in – flow out

In order to obtain a mathematical model for water balance in the great lakes it is essential to observe and evaluate the individual change in volume of the lakes individually. (parameters explained in the next section)

LAKE SUPERIOR

Flow In = $P_1 + R_1 + D_1$
Flow Out = $F_{o1} + E_1$

$\frac{dV_1}{dt} = P_1 + R_1 + D_1 - F_{o1} - E_1$ m^3/month

LAKE MICHIGAN-HURON

Flow In = $F_{o1} + P_2 + R_2$
Flow Out = $F_{o2} + D_2 + E_2$

$\frac{dV_2}{dt} = F_{o1} + P_2 + R_2 - F_{o2} - D_2 - E_2$ m^3/month

LAKE ERIE

Flow In = $F_{o2} + P_3 + R_3$
Flow Out = $F_{o3} + D_3 + E_3$

$\frac{dV_3}{dt} = F_{o2} + P_3 + R_3 - F_{o3} - D_3 - E_3$ m^3/month

LAKE ONTARIO

Flow In = $F_{o3} + P_4 + R_4 + D_3$
Flow Out = $F_{o4} + E_3$

$\frac{dV_4}{dt} = F_{o3} + P_4 + R_4 + D_3 - F_{o4} – E_{4.}$ m^3/month

The trend from the individual models of the lakes observed reveals that precipitation and run off are always a part of inflow and evaporation of outflow. Also, the outflow of one lake is the inflow of the next lake so $F_{in} = F_{o(i-1)}$. Since no generalisation can be made of the diversions flowing in and out of the lakes the flow chart can be used to analyse its progression.

Observing the trends from the individual models of the lakes a generalised equation for the water balance of the great lakes.

$$\boxed{\frac{dV_i}{dt} = \mathbf{F_{o(i-1)} + P_i + R_i - F_{oi} - E_i \pm D_i}} \quad m^3/month \qquad ....(i)$$

We need to obtain an expression for the volume with respect to time (V(t)). This can be done by integrating the equation (i)
Eq (i) can be re-written as

$$dV_i = (F_{o(i-1)} + P_i + R_i - F_{oi} - E_i \pm D_i) \, dt$$

Integrating both sides

$$\Rightarrow \int dV_i = \int (F_{o(i-1)} + P_i + R_i - F_{oi} - E_i \pm D_i) \, dt$$

$$\Rightarrow V_i(t) = (F_{o(i-1)} + P_i + R_i - F_{oi} - E_i \pm D_i) \, t + C$$

where C = constant of Integration

At t=0 , the volume, i.e., the initial volume = V_{io}
Replacing these conditions into the above equation we get

$$C = V_o$$

$$\Rightarrow \mathbf{V_i(t) = (F_{o(i-1)} + P_i + R_i - F_{oi} - E_i \pm D_i) \, t + V_{io}} \; m^3$$

In order to bring about this model certain assumptions were made for simplification. Firstly, in order to convert seconds into months, all months were assumed to have 30 days. Also, in order to convert mm/month into m^3/month, only the present-day surface areas of the lakes were used for all of the data.

The assumptions made for simplification might cause slight inaccuracies in the data. However, these inaccuracies will not cause any significant change in the results obtained. But it should be noted that the present-day surface area was used hence, not taking into account the changing surface area of the lakes through the years. Also, for the conversion into months for 30 days was used it doesn't take into consideration months actually have more and less days.

PARAMETRISATION AND IMPLEMENTATION

The given data set describes the various sources of inflow and outflow of the great lakes – Lake Superior, Lake Michigan, Lake Huron, Lake Erie And Lake Ontario. These are defined as follows

The parameters used are:
P = Precipitation (addition in volume due lake)
E = Evaporation (loss of water as water vapour)
R = Water Runoff (water from surrounding land flowing into the lake)
D = Diversions (transfer of water through tubes or canals)
F_o = Flow Out (water flowing out of the lake)

Subscripts to the above-mentioned parameters are defined as
1 – Lake Superior
2 – Lake Michigan-Huron
3 – Lake Erie
4 – Lake Ontario

For the sake of simplification St. Claire Lake has not been considered and Lake Michigan and Lake Huron are considered as a single lake.

In the dataset, precipitation, evaporation and run off were given in the units 'mm over the respective lake surface area (m^2) per month i.e. mm/month. These were converted into m^3/month [A1]. Diversions and Flow Out were given in m^3/second. Since the per second changes were too small to be meaningful these were converted into m^3/month [A2].

The following table shows the present surface areas of the lakes used as well as their volumes as of 1950:

LAKE	SURFACE AREA (m^2) [1]	VOLUME (m^3) [2]
Lake Superior	8.21×10^{10}	1.22×10^{13}
Lake Michigan – Huron	1.174×10^{11}	8.45×10^{12}
Lake Erie	2.567×10^{10}	4.88×10^{11}
Lake Ontario	1.901×10^{10}	1.64×10^{12}

Table 1: Present Surface Areas and Volumes of the Great Lakes in 1950

The volume represented above is used as the initial volume V_{io}

The time step chosen for the model is one month as it was deemed most appropriate in order to adequately analyse significant changes in the volume. This is because daily changes would not enable significant analysis and the yearly changes are not significant since most of the changes occur within a year.

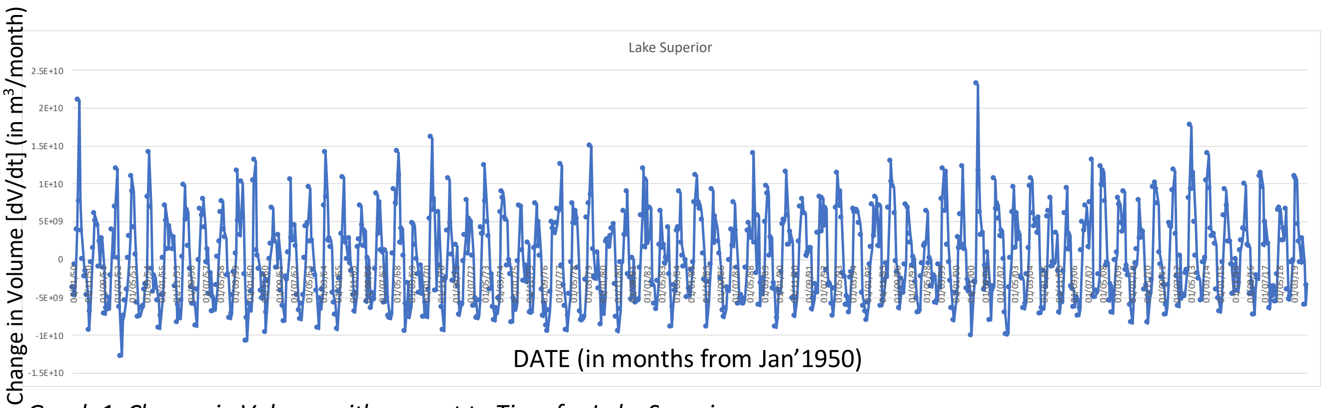
DIMENSIONALLY ANALYSIS of Eq (i)

Parameter	Unit	Dimension
dV_i/dt	m^3/month	L^3/T
F_i	m^3/month	L^3/T
P_i	m^3/month	L^3/T
R_i	m^3/month	L^3/T
E_i	m^3/month	L^3/T
D_i	m^3/month	L^3/T

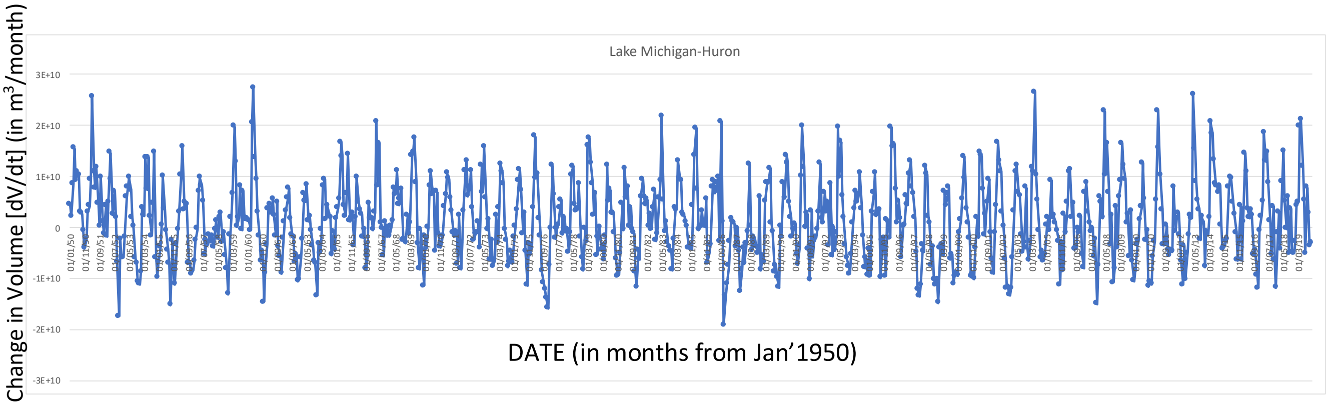
Thus, since the dimensions on the right side of the equation are same as the left side, it is dimensionally homogeneous.

RESULTS

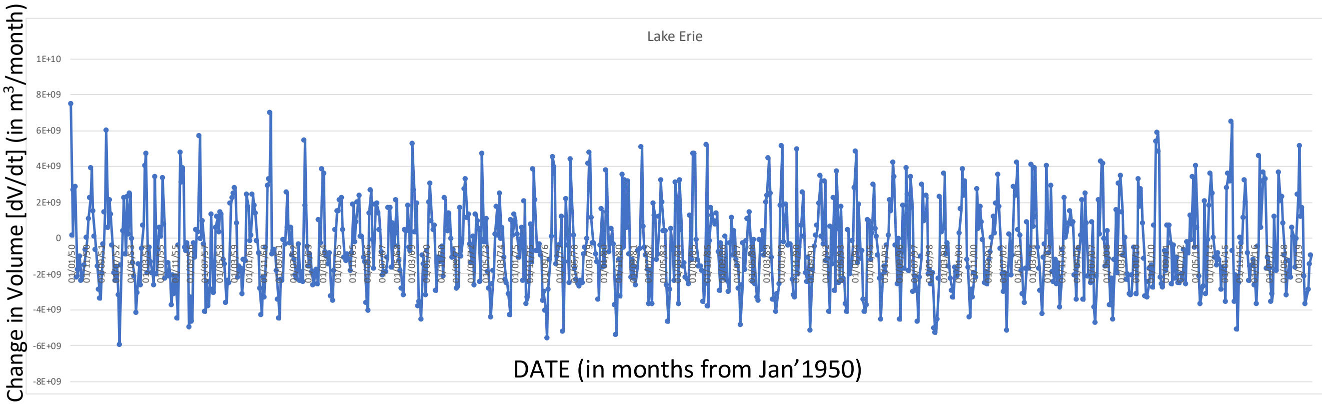
The provided data, through the equations derived, was used to plot the change in volume of the lakes through the course of time. This is shown by the following graphs.



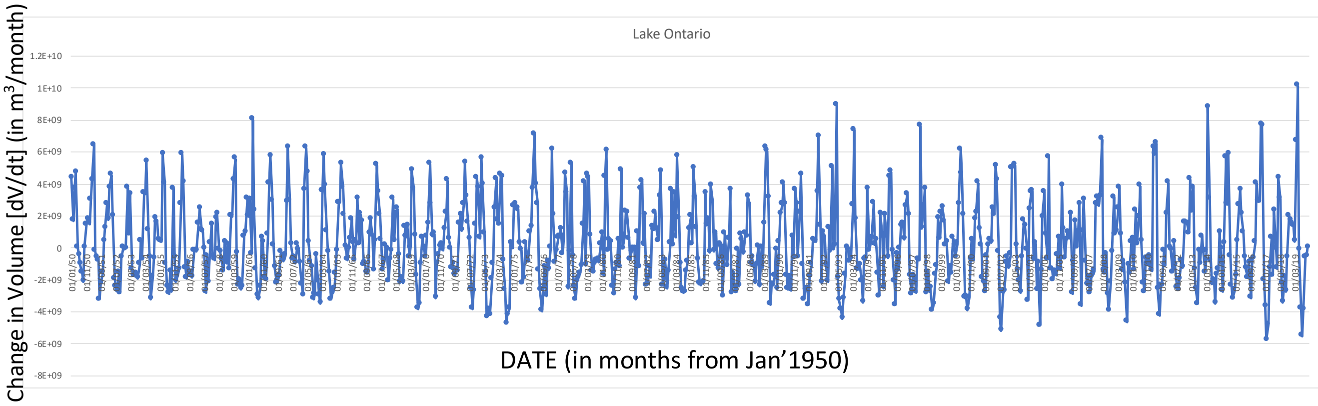
Graph 1. Change in Volume with respect to Time for Lake Superior



Graph 2. Change in Volume with respect to Time for Lake Michigan-Huron



Graph 3. Change in Volume with respect to Time for Lake Erie



Graph 4. Change in Volume with respect to Time for Lake Ontario

Alternatively, we can also evaluate the total inflows and outflows of the Lake to study the overall flow of water through the lake.

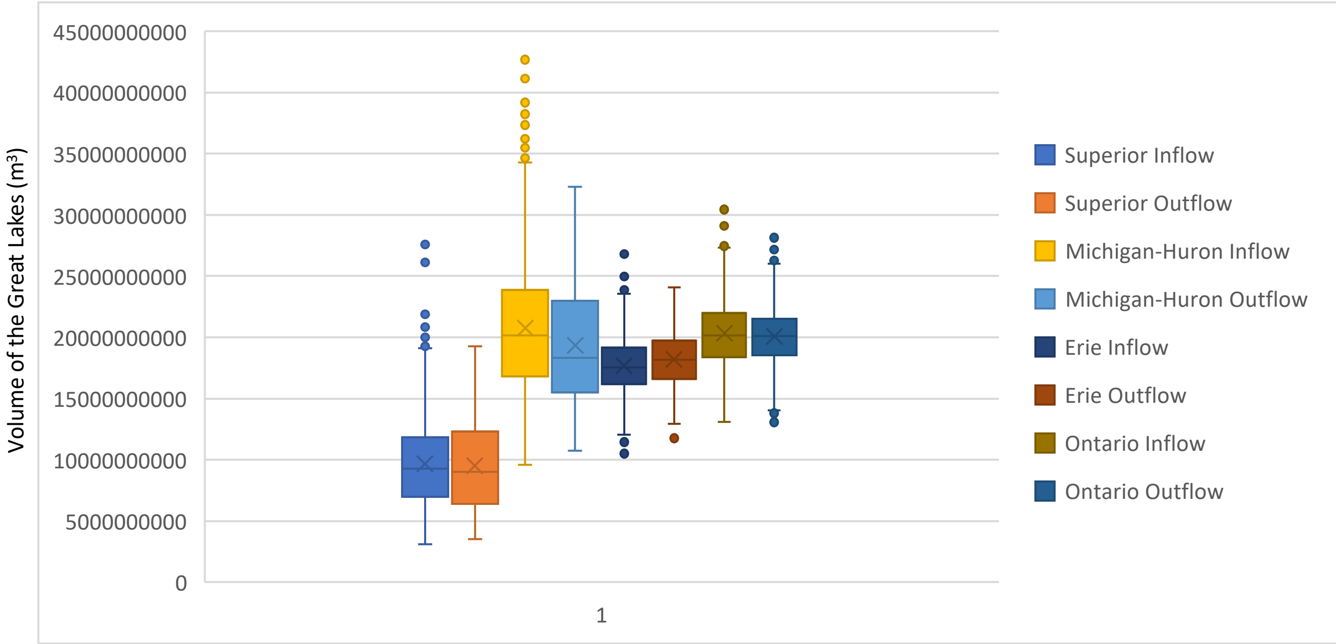
MEAN[A4.] (m^3/month)							
Superior		Michigan-Huron		Erie		Ontario	
INFLOW	OUTFLOW	INFLOW	OUTFLOW	INFLOW	OUTFLOW	INFLOW	OUTFLOW
9.674E+09	9.499E+09	2.076E+10	1.94E+10	1.77E+10	1.82E+10	2.03E+10	2.01E+10

Table 2: Mean Value of Inflow and Outflow of the Great Lakes

STANDARD DEVIATION[A5.]							
Superior		Michigan-Huron		Erie		Ontario	
INFLOW	OUTFLOW	INFLOW	OUTFLOW	INFLOW	OUTFLOW	INFLOW	OUTFLOW
3.6E+09	3.536E+09	5.409E+09	4.55E+09	2.25E+09	2.14E+09	2.73E+09	2.3E+09

Table 3: Standard Deviation of Inflow and Outflow of the Great Lakes

Plotting a box plot graph using this data enables us to understand how the volume changes and the variation in this change through the years.



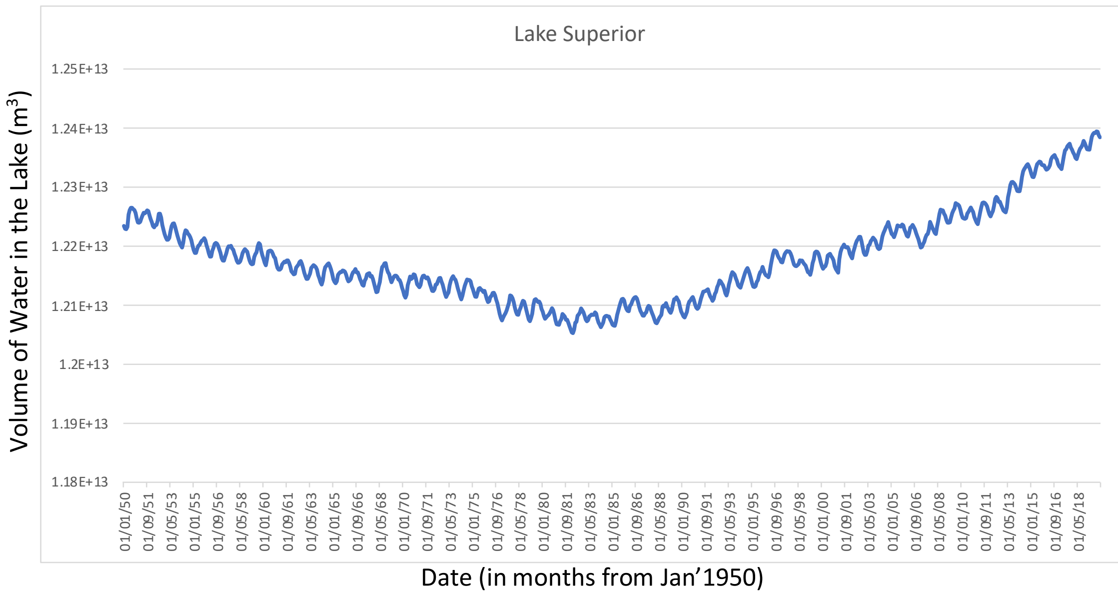
Graph 5. Box Plot representing mean and standard deviation of the inflows and outflows of the Great Lakes

In order to find V(t), we need to numerically integrate the expression for dV_i/dt. To do this we use Euler's method which states

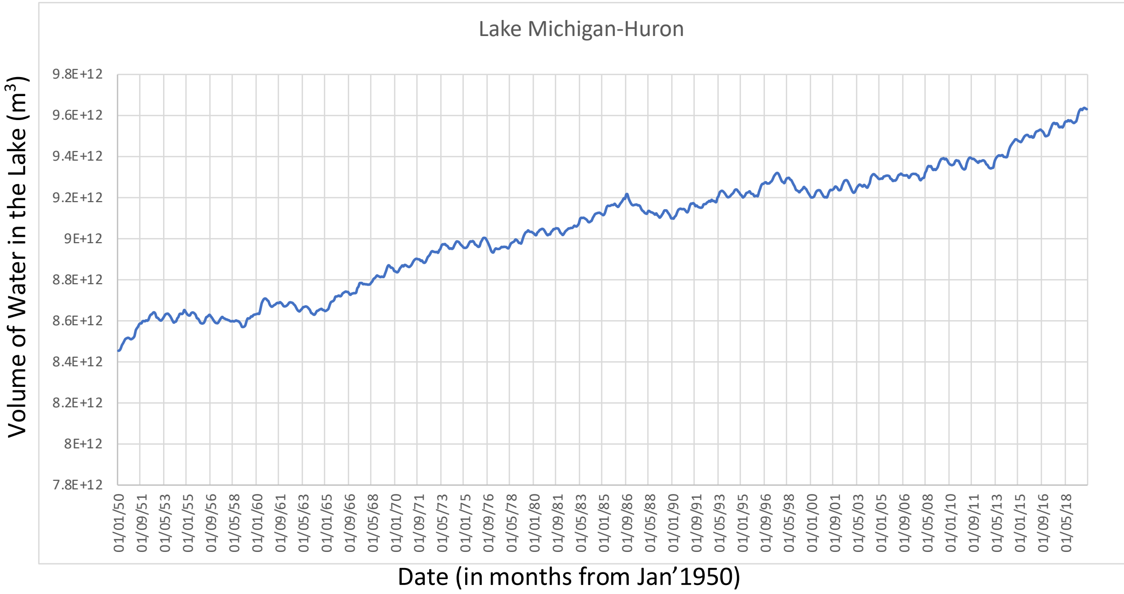
$$y(x + \Delta x) \approx y(x) + \frac{dy}{dx} \,.\, \Delta x$$

We know for each lake, the initial volumes of the lakes V_{oi} and the change in volume dV_i/dt. We have also chosen a timestep Δx = 1 month, we find the the value of $V_i(t)$ through numerical integration of dV_i/dt .

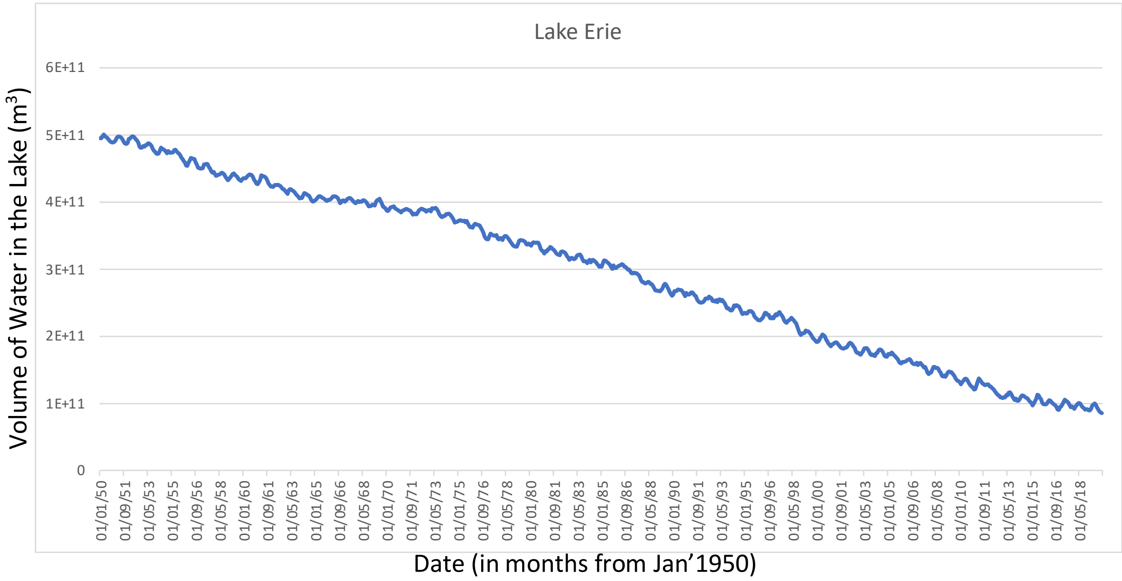
We calculate the cumulative sum of dV_i/dt through the dataset and add it to initial volume V_{oi}. [A6.]



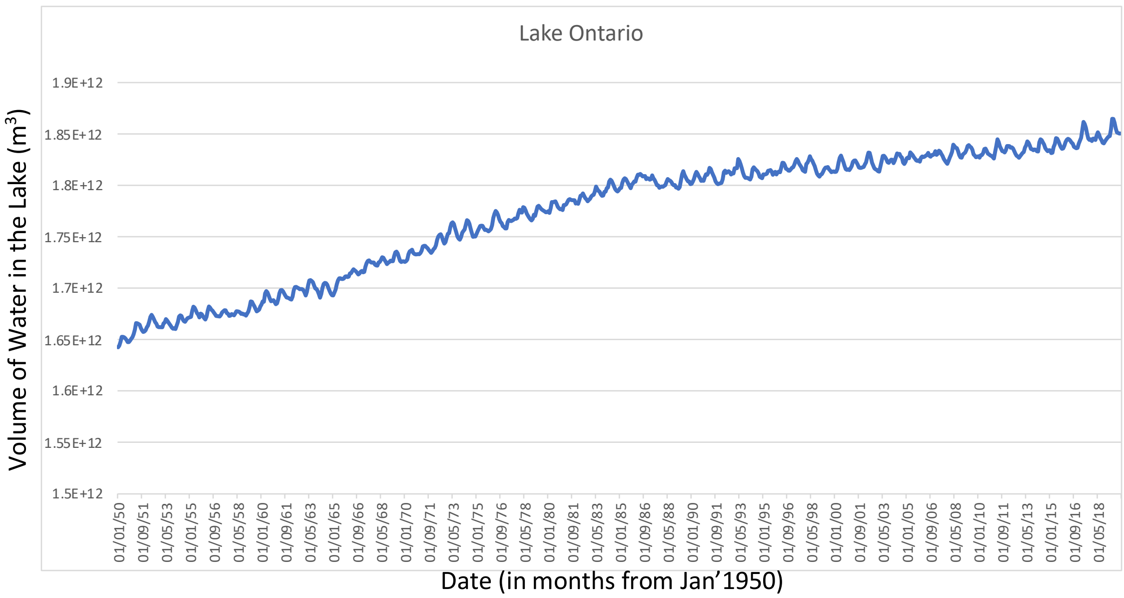
Graph 6. Volume of the Lake with respect to Time for Lake Superior



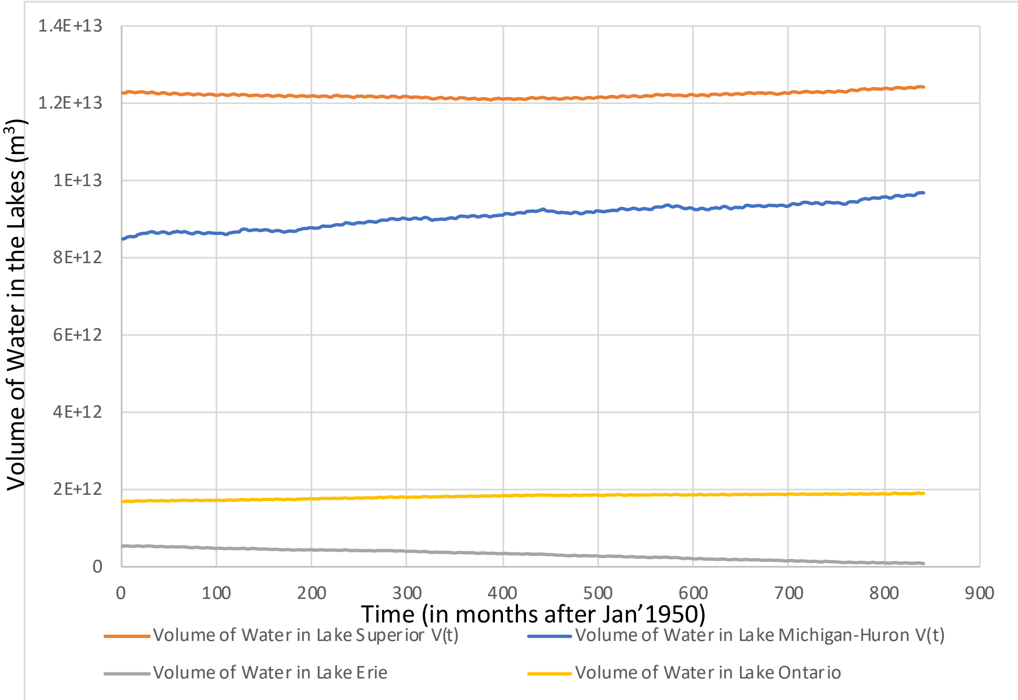
Graph 7. Volume of the Lake with respect to Time for Lake Michigan-Huron



Graph 8. Volume of the Lake with respect to Time for Lake Erie



Graph 9. Volume of the Lake with respect to Time for Lake Ontario



Graph 10. Volume of the Great Lakes with respect to Time

From the graphs shown above, it can be observed that Lake Superior (with maximum volume) has an almost steady volume through the years. Lake Michigan-Huron show an unsteady trend of slow increase in volume through the years. Lake Erie shows an almost linear decline in volume while Lake Ontario shows a subtle linear incline.

ENGINEERING SOLUTION

The results obtained in the previous section helped us analyse how the volumes of the lakes changes over time. It was analysed in *Graph 8* and *10* that the water levels in Lake Erie have decreased since 1950.

We need to obtain $E\,[dV_3/dt] \approx 0$

We have from before, $\frac{\mathbf{dV_3}}{\mathbf{dt}} = F_{o2} + P_3 + R_3 - F_{o3} - D_3 - E_3$

$E\,[dV_3/dt] = E\,[F_{o2} + P_3 + R_3 - F_{o3} - D_3 - E_3]$

Replacing with $E\,[dV_3/dt] = 0$

- ⇨ $E\,[F_{o2} + P_3 + R_3 - F_{o3} - D_3 - E_3] = 0$
- ⇨ $E\,[F_{o3}]_{new} = E\,[F_{O2} + P_3 + R_3 - D_3 - E_3]$
- ⇨ $E\,[F_{o3}]_{new} = 15244287626\ m^3/month$

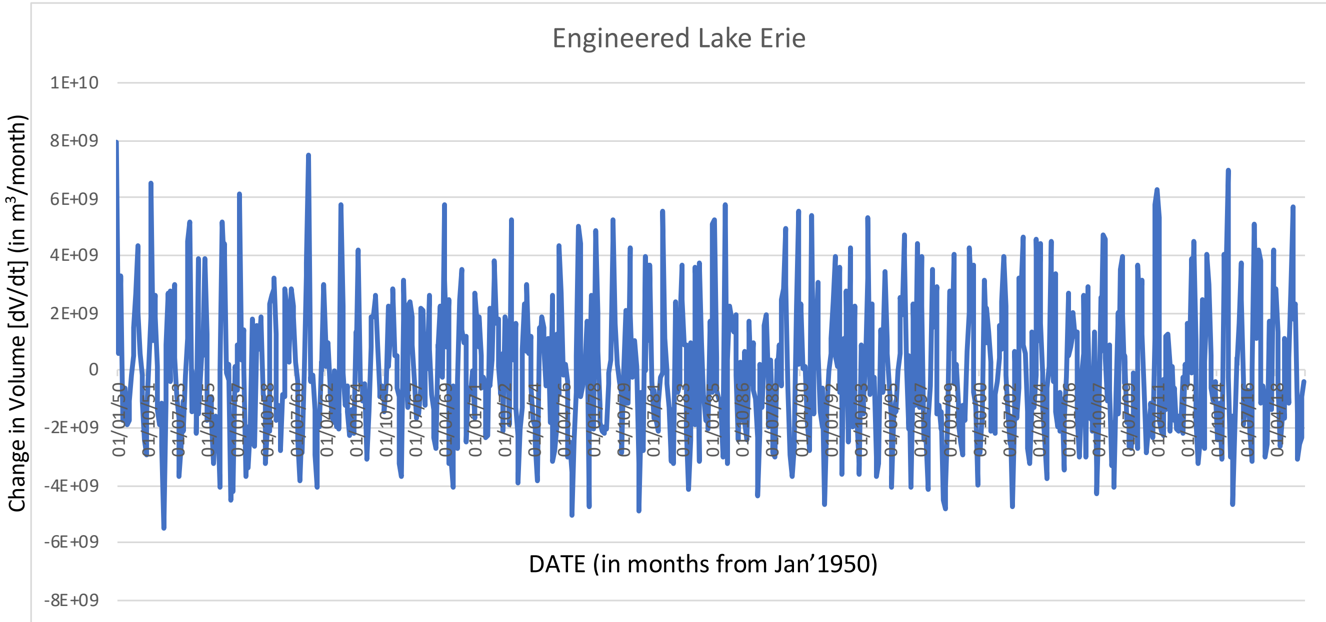
We also have $E\,[F_{o3}]_{old} = 15722679898\ m^3/month$

Thus, we can calculate the Scale Factor = $E\,[F_{o3}]_{new} / E\,[F_{o3}]_{old}$

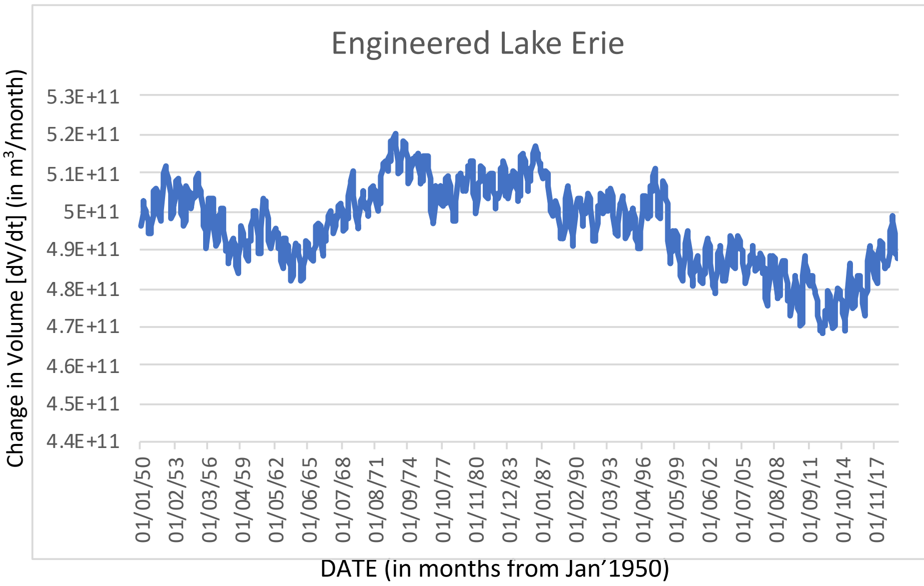
⇨ Scale Factor = 0.969573109

In order to get the desired outflow, we multiply the data $[F_{o3}]_{old}$ by the Scale Factor to get $F_{o3}^{desired}$

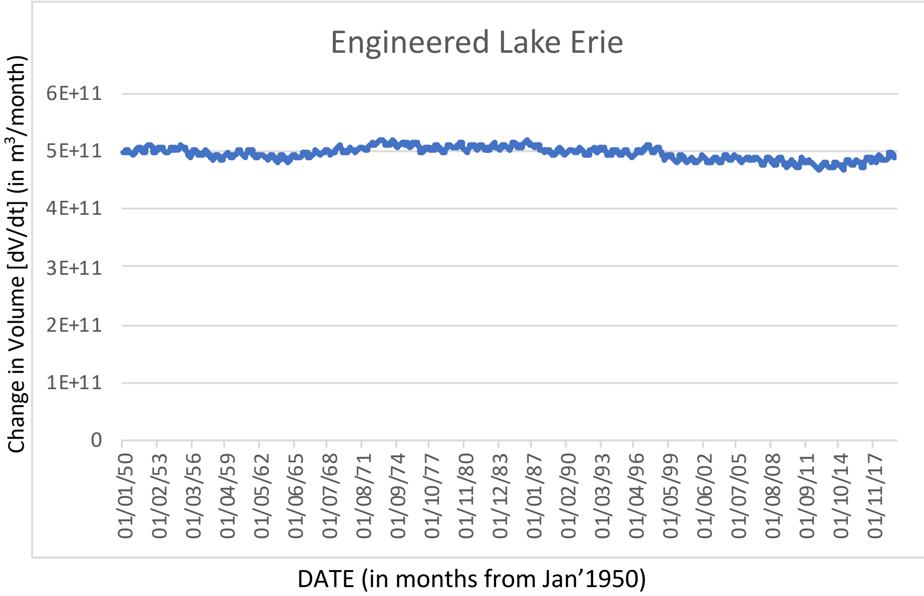
We calculate the new dV_3/dt and hence the new $V_3(t)$ as before to compare results



Graph 11. Engineered Change in Volume with respect to Time for Lake Erie



Graph 12. Engineered Volume of the Lake with respect to Time for Lake Erie (zoomed in)



Graph 13. Engineered Volume of the Lake with respect to Time for Lake Erie (zoomed out)

These graphs show the change in volume is negligible. *Graph 13* shows an almost straight line indicating almost steady-state behaviour in the flow of Lake Erie.

It is certain that a change in the outflow of Lake Erie would change the inflow for Lake Ontario. Hence, using these new inflows from Lake Erie, we repeat the same process for Lake Ontario We need to obtain E $[dV_3/dt] \approx 0$

We have from before, $\frac{\mathbf{dV_4}}{\mathbf{dt}} = F_{o3} + P_4 + R_4 + D_3 - F_{o4} - E_4$

$E\,[dV_4/dt] = E\,[F_{o3} + P_4 + R_4 + D_3 - F_{o4} - E_4]$

Replacing with $E\,[dV_4/dt] = 0$

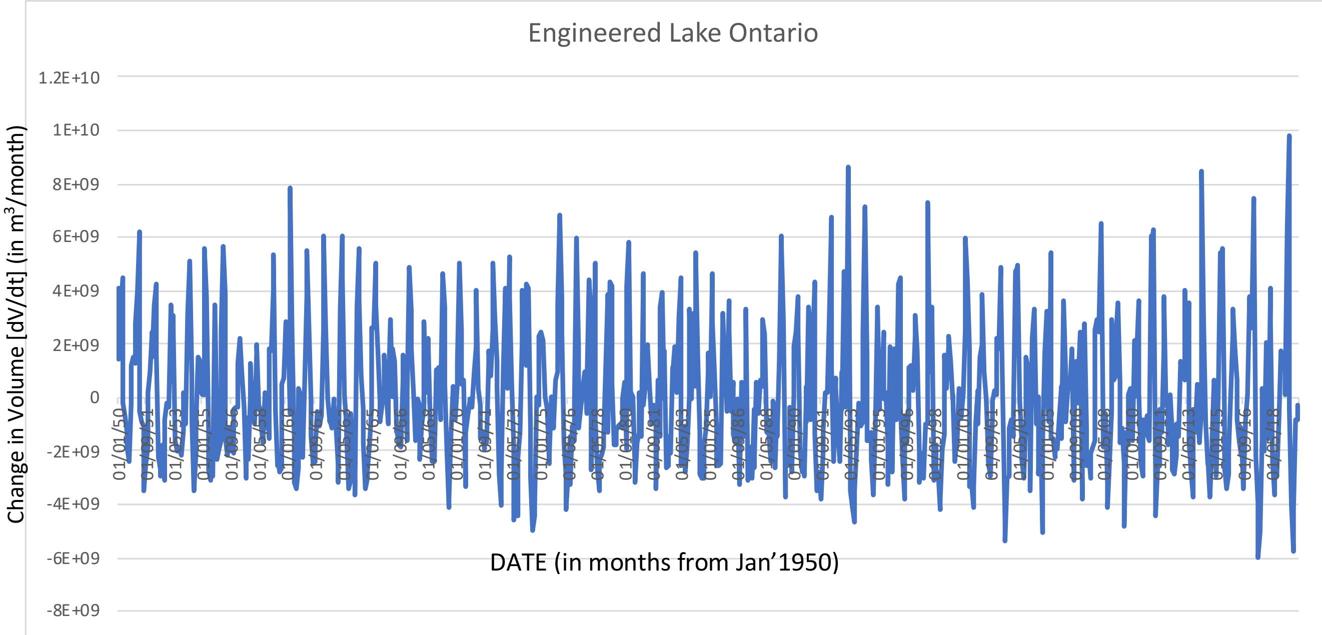
- ⇨ $E\,[F_{o3} + P_4 + R_4 + D_3 - F_{o4} - E_4] = 0$
- ⇨ $E\,[F_{o4}]_{new} = E\,[F_{o3} + P_4 + R_4 + D_3 - E_4]$
- ⇨ $E\,[F_{o4}]_{new} = 18803382525\ m^3/month$

We also have $E\,[F_{o4}]_{old} = 19028876277\ m^3/month$

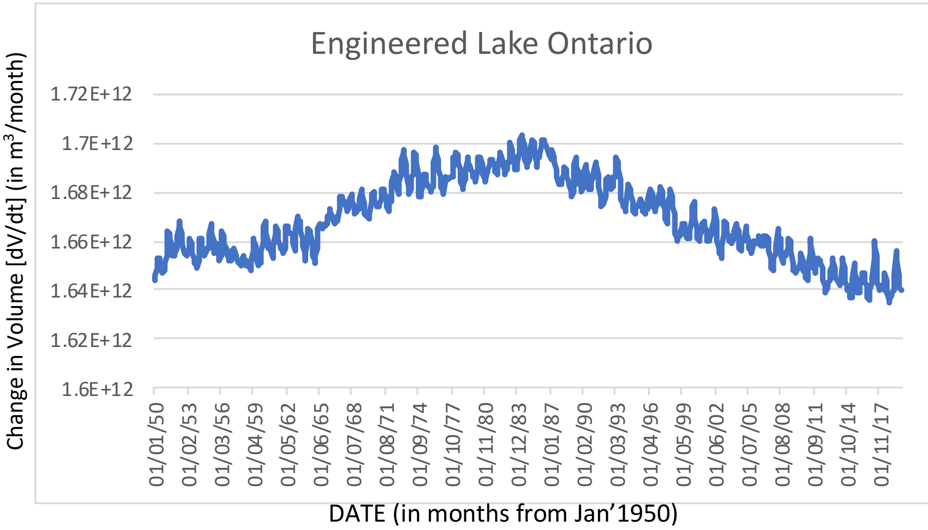
Thus, we can calculate the Scale Factor = $E\,[F_{o4}]_{new} / E\,[F_{o4}]_{old}$
- ⇨ Scale Factor = 0.988149918

In order to get the desired outflow, we multiply the data $[F_{o4}]_{old}$ by the Scale Factor to get $F_{o4}^{desired}$

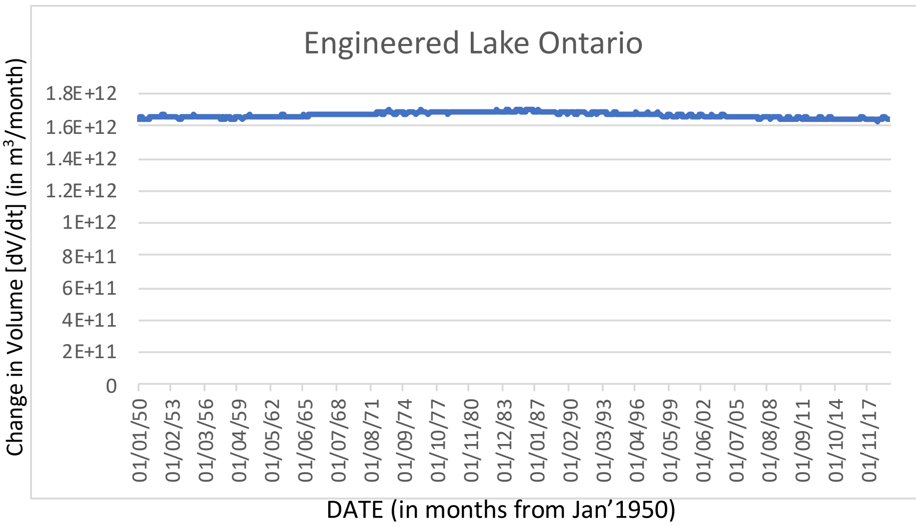
We calculate the new dV_4/dt and hence the new $V_4(t)$ as before to compare results



Graph 14. Engineered Change in Volume with respect to Time for Lake Ontario



Graph 15. Engineered Volume of the Lake with respect to Time for Lake Ontario (zoomed in)



Graph 16. Engineered Volume of the Lake with respect to Time for Lake Erie (zoomed out)

These graphs show the change in volume is negligible. *Graph 16* shows an almost straight line indicating almost steady-state behaviour in the flow of Lake Erie.

To achieve these results, the proposed solution to accomplish the two new flow rates i.e., $F_{o3}^{desired}$ and $F_{o4}^{desired}$ is to construct two dams. The first should be constructed on the Niagra River to control outflows of Lake Erie by allowing only 96.96% of the water to flow through. This would affect inflows into Lake Ontario. The second should be constructed on St. Lawrence River to control outflows of Lake Ontario by allowing only 98.81% of the water to flow through.

REFERENCES

1. Areas and Volumes of the Great Lakes. (2019). In: *Encyclopædia Britannica*. [online] Available at: https://www.britannica.com/topic/Areas-and-Volumes-of-the-Great-Lakes-1800353.
2. Wilcox, D., Thompson, T., Booth, R. and Nicholas, J.R. (2007). *Lake-Level Variability and Water Availability in the Great Lakes*. [online] . Available at: https://pubs.usgs.gov/circ/2007/1311/pdf/circ1311_web.pdf.

APPENDIX

A1. <u>Conversion of mm/month into m^3/month</u>

(x_i). (SurfaceArea*)/1000

where x_i is a given element from the dataset

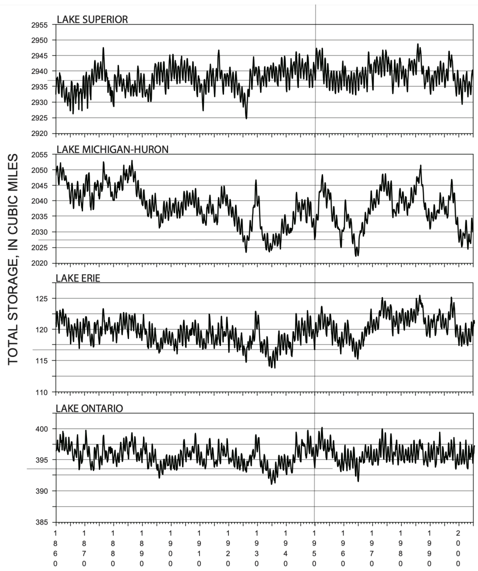
*from *Table 1*

A2. <u>Conversion of m^3/second into m^3/month</u>

(x_i).(30)(24)(3600)

where x_i is a given element from the dataset

A3. The volume was approximately taken from a table from the source mentioned [2] was given in cubic miles and was converted into cubic metres



LAKE	Volume (cubic miles)	Volume (cubic metres)
Lake Superior	2936	1.22×10^{13}
Lake Michigan-Huron	2027.5	8.45×10^{12}
Lake Erie	117	4.88×10^{11}
Lake Ontario	393	1.64×10^{12}

A4. Mean

Calculated using command AVERAGE(number1, number2, ...) on excel

$\overline{X} = \frac{1}{N}\Sigma(X_i)$

A5. Standard Deviation

Calculated using command STDEV(number1, number2, ...) on excel

$$SD = \sqrt{\frac{1}{N}\Sigma(Xi - X)^2}$$

A6. Calculating $V_i(t)$ using Eulers' method

Demonstating for Lake Superior (same process used for all Lakes)

First calculate dV/dt

Column Formula
(= G + E + P – H – C)

A	B	C	D	E	F	G	H	I	J
LAKE SUPERIOR									
Date		EVAPORATION	PRECIPITATION	RUN OFF		DIVERSIONS	FLOW OUT		dV/dt
01/01/50		8714915000	7057316000	2203564000		368012160	4625216640		-3711239480
01/02/50		4372646000	2330819000	1791422000		323326080	4863628800		-4790707720
01/03/50		2561520000	3914528000	2479420000		283694400	4843048320		-726925920
01/04/50		1816052000	5420242000	4824196000		253549440	4846728960		3835206480
01/05/50		117403000	7593429000	17881380000		654842880	5002560000		21009688880
01/06/50		-268467000	8159919000	6687045000		175322880	7670505600		7620248280
01/07/50		-324295000	7343024000	4999890000		175633920	9048153600		3794689320
01/08/50		193756000	5882465000	3408792000		158863680	9271402560		-15037880
01/09/50		1053343000	4985112000	2468747000		513552960	9205747200		-2291678240

Then calculate Cumulative dV/dt

J	K
dV/dt	CUMULATIVE
-3711239480	-3711239480
-4790707720	=K3+J4
-726925920	-9228873120
3835206480	-5393666640
21009688880	15616022240
7620248280	23236270520

→

J	K
dV/dt	CUMULATIVE
-3711239480	-3711239480
-4790707720	-8501947200
-726925920	=K4+J5
3835206480	-5393666640
21009688880	15616022240
7620248280	23236270520
3794689320	27030959840
-15037880	27015921960
-2291678240	24724243720

Finally, to calculate V(t), use the initial volume V_{io} from *Table 1* and add to cumulative dV/dt

J	K	L
dV/dt	CUMULATIVE	VOLUME
-3711239480	-3711239480	=12238000000000+K3
-4790707720	-8501947200	1.22295E+13
-726925920	-9228873120	1.22288E+13
3835206480	-5393666640	1.22326E+13
21009688880	15616022240	1.22536E+13
7620248280	23236270520	1.22612E+13
3794689320	27030959840	1.2265E+13
-15037880	27015921960	1.2265E+13
-2291678240	24724243720	1.22627E+13
-1537838800	23186404920	1.22612E+13
[illegible]	[illegible]	[illegible]